

# A FRACTIONAL HELLY THEOREM FOR BOXES

I. BÁRÁNY, F. FODOR, A. MARTÍNEZ-PÉREZ, L. MONTEJANO, D. OLIVEROS,  
AND A. PÓR

*This paper is dedicated to Javier Bracho on occasion of his sixtieth birthday.*

**ABSTRACT.** Let  $\mathcal{F}$  be a family of  $n$  axis-parallel boxes in  $\mathbb{R}^d$  and  $\alpha \in (1-1/d, 1]$  a real number. There exists a real number  $\beta(\alpha) > 0$  such that if there are  $\alpha \binom{n}{2}$  intersecting pairs in  $\mathcal{F}$ , then  $\mathcal{F}$  contains an intersecting subfamily of size  $\beta n$ . A simple example shows that the above statement is best possible in the sense that if  $\alpha \leq 1 - 1/d$ , then there may be no point in  $\mathbb{R}^d$  that belongs to more than  $d$  elements of  $\mathcal{F}$ .

## 1. INTRODUCTION AND RESULTS

According to the classical theorem of Helly [1], if every  $d + 1$ -element subfamily of a finite family of convex sets in  $\mathbb{R}^d$  has nonempty intersection, then the entire family has nonempty intersection. Although the number  $d + 1$  in Helly's theorem cannot be lowered in general, it can be reduced for some special families of convex sets. For example, if any two elements in a finite family of axis-parallel boxes in  $\mathbb{R}^d$  intersect, then all members of the family intersect, cf. [2].

Katchalski and Liu [7] proved the following generalization of Helly's theorem for the case when not all but only a fraction of  $d + 1$ -element subfamilies have a nonempty intersection in a family of convex sets.

**Fractional Helly Theorem.** (Katchalski and Liu [7]) *Assume that  $\alpha \in (0, 1]$  is a real number and  $\mathcal{F}$  is a family of  $n$  convex sets in  $\mathbb{R}^d$ . If at least  $\alpha \binom{n}{d+1}$  of the  $(d+1)$ -tuples of  $\mathcal{F}$  intersect, then  $\mathcal{F}$  contains an intersecting subfamily of size  $\frac{\alpha}{d+1}n$ .*

The bound on the size of the intersecting subfamily was later improved by Kalai [6] from  $\frac{\alpha}{d+1}n$  to  $(1 - (1 - \alpha)^{1/(d+1)})n$ , and this bound is best possible.

In this paper, we study the fractional behaviour of finite families of axis-parallel boxes, or boxes for short. We note that the boxes can be either open or closed, our statements hold for both cases. Our aim is to prove a statement similar to the Fractional Helly Theorem.

The intersection graph  $\mathcal{G}_{\mathcal{F}}$  of a finite family  $\mathcal{F}$  of boxes is a graph whose vertex set is the set of elements of  $\mathcal{F}$ , and two vertices are connected by an edge in  $\mathcal{G}_{\mathcal{F}}$  precisely when the corresponding boxes in  $\mathcal{F}$  have nonempty intersection.

Recall that for two integers  $n \geq m \geq 1$ , the Turán-graph  $\mathcal{T}(n, m)$  is a complete  $m$ -partite graph on  $n$  vertices in which the cardinalities of the  $m$  vertex classes are as close to each other as possible. Let  $t(n, m)$  denote the number of edges of the Turán graph  $\mathcal{T}(n, m)$ . It is known that  $t(n, m) \leq (1 - \frac{1}{m})\frac{n^2}{2}$ , and equality holds if  $m$  divides  $n$ . Furthermore,

$$\lim_{n \rightarrow \infty} \frac{t(n, m)}{\frac{n^2}{2}} = 1 - \frac{1}{m}. \quad (1)$$

For more information on the properties of Turán graphs see, for example, the book of Diestel [3].

The following example shows that we cannot hope for a statement for boxes that is completely analogous to the Fractional Helly Theorem.

**Example 1.** Let  $n \geq d + 1$  and  $m, k \geq 0$  be integers such that  $n = md + k$  and  $0 \leq k \leq d - 1$ . Let  $n_1, \dots, n_d$  be positive integers with  $n = n_1 + \dots + n_d$  and  $n_i = \lceil \frac{n}{d} \rceil$  for  $1 \leq i \leq k$  and  $n_i = \lfloor \frac{n}{d} \rfloor$  for  $k + 1 \leq i \leq d$ . For  $1 \leq i \leq d$ , consider  $n_i - 1$  hyperplanes orthogonal to the  $i$ th coordinate direction. These hyperplanes cut  $\mathbb{R}^d$  into  $n_i$  pairwise disjoint open slabs  $B'_{ij}, j = 1, \dots, n_i$ . Let  $C$  be a large open axis-parallel box that intersects each slab and let  $\mathcal{F}_i$  consist of the open boxes  $B_{ij} = C \cap B'_{ij}$ . Define  $\mathcal{F}$  as the union of the  $\mathcal{F}_i$ .

This way we have obtained a family  $\mathcal{F}$  of  $n$  boxes with the property that two elements of  $\mathcal{F}$  intersect exactly if they belong to different  $\mathcal{F}_i$ . The intersection graph of  $\mathcal{F}$  is  $\mathcal{T}(n, d)$  and thus the number of intersecting pairs in  $\mathcal{F}$  is  $t(n, d)$ . However, there is no point of  $\mathbb{R}^d$  that belongs to any  $d + 1$ -element subfamily of  $\mathcal{F}$ . Thus, (1) shows that in a fractional Helly-type statement for boxes, the percentage  $\alpha$  has to be greater than  $1 - \frac{1}{d}$ .

Let  $n \geq k \geq d$  and let  $T(n, k, d)$  denote the maximal number of intersecting pairs in a family  $\mathcal{F}$  of  $n$  boxes in  $\mathbb{R}^d$  with the property that no  $k + 1$  boxes in  $\mathcal{F}$  have a point in common.

**Theorem 1.** *With the above notation,*

$$T(n, k, d) < \frac{d-1}{2d}n^2 + \frac{2k+d}{2d}n.$$

It is quite easy to precisely determine  $T(n, k, d)$  when  $d = 1$ :

**Proposition 1.**  $T(n, k, 1) = (k-1)n - \binom{k}{2}$ .

Theorem 1 directly implies the following corollary.

**Corollary 1.** *Assume that  $\varepsilon > 0$  is a real number and  $\mathcal{F}$  is a family of  $n$  boxes in  $\mathbb{R}^d$ . If at least  $(\frac{d-1}{2d} + \varepsilon)n^2$  pairs of  $\mathcal{F}$  intersect, then  $\mathcal{F}$  contains an intersecting subfamily of size  $dn\varepsilon - \frac{d}{2} + 1$ .*

The proof of Corollary 1 is given in Subsection 2.2. Corollary 1 yields the next theorem, which is our main result.

**Fractional Helly Theorem for boxes.** *For every  $\alpha \in (1 - \frac{1}{d}, 1]$  there exists a real number  $\beta(\alpha) > 0$  such that, for every family  $\mathcal{F}$  of  $n$  boxes in  $\mathbb{R}^d$ , if an  $\alpha$  fraction of pairs are intersecting in  $\mathcal{F}$ , then  $\mathcal{F}$  has an intersecting subfamily of cardinality at least  $\beta n$ .*

Kalai's lower bound  $\beta(\alpha) = 1 - (1 - \alpha)^{1/(d+1)}$  for the size of the intersecting subfamily in the fractional Helly theorem yields that if  $\alpha \rightarrow 1$ , then  $\beta(\alpha) \rightarrow 1$  as well. The same holds for families of parallel boxes as stated in the following theorem.

**Theorem 2.** *Let  $\mathcal{F}$  be a family of  $n$  boxes in  $\mathbb{R}^d$ , and let  $\alpha \in (1 - \frac{1}{d^2}, 1]$  be a real number. If at least  $\alpha \binom{n}{2}$  pairs of boxes in  $\mathcal{F}$  intersect, then there exists a point that belongs to at least  $(1 - d\sqrt{1 - \alpha})n$  elements of  $\mathcal{F}$ .*

Simple calculations show that Corollary 1 does not imply Theorem 2 so we provide a separate proof for it in Section 2.

## 2. PROOFS

**2.1. Proof of Theorem 1.** It is enough to prove that if no  $k + 1$  elements of  $\mathcal{F}$  have a point in common, then there are at least  $\frac{n^2 - 2(k+d)n}{2d}$  non-intersecting pairs. We may assume by standard arguments that the boxes in  $\mathcal{F}$  are all open, so  $B \in \mathcal{F}$  is of the form  $B = (a_1(B), b_1(B)) \times \cdots \times (a_d(B), b_d(B))$ . We assume without loss of generality that all numbers  $a_i(B), b_i(B)$  ( $B \in \mathcal{F}$ ) are distinct. For  $B \in \mathcal{F}$  we define  $\deg B$  to be the number of boxes in  $\mathcal{F}$  that intersect  $B$ .

We prove Theorem 1 by induction on  $n$ . The starting case  $n = k$  is simple since then  $\frac{n^2 - 2(k+d)n}{2d} < 0$ . In the induction step  $n - 1 \rightarrow n$  we consider two cases.

**Case 1.** When there is a box  $B$  with  $\deg B \leq (1 - \frac{1}{d})n + \frac{2k+1}{2d}$ .

By induction, we have at least  $\frac{(n-1)^2 - 2(k+d)(n-1)}{2d}$  non-intersecting pairs after removing  $B$  from  $\mathcal{F}$ . Since  $B$  is involved in at least  $(n-1) - (1 - \frac{1}{d})n - \frac{2k+1}{2d}$  non-intersecting pairs, there are at least

$$\frac{(n-1)^2 - 2(k+d)(n-1)}{2d} - 1 + \frac{n}{d} - \frac{2k+1}{2d} = \frac{n^2 - 2(k+d)n}{2d}$$

non-intersecting pairs in  $\mathcal{F}$ , indeed.

**Case 2.** For every  $B \in \mathcal{F}$   $\deg B \geq (1 - \frac{1}{d})n + \frac{2k+1}{2d}$ .

We show by contradiction that this cannot happen which finishes the proof.

We define  $d$  distinct boxes  $B_1, \dots, B_d \in \mathcal{F}$  the following way. Set

$$c_1 = \min\{b_1(B) : B \in \mathcal{F}\}$$

and define  $B_1$  via  $c_1 = b_1(B_1)$ . The box  $B_1$  is uniquely determined as all  $b_1(B)$  are distinct numbers. Assume now that  $i < d$  and that the numbers  $c_1, \dots, c_{i-1}$ , and boxes  $B_1, \dots, B_{i-1}$  have been defined. Set

$$c_i = \min\{b_i(B) : B \in \mathcal{F} \setminus \{B_1, \dots, B_{i-1}\}\}$$

and define  $B_i$  via  $c_i = b_i(B_i)$  which is unique, again.

Let  $\mathcal{F}' = \mathcal{F} \setminus \{B_1, \dots, B_d\}$ . We partition  $\mathcal{F}'$  into  $d + 2$  parts. Let  $\mathcal{F}_0$  be the set of all boxes of  $\mathcal{F}'$  that intersect every  $B_i$ . For  $i = 1, \dots, d$  let  $\mathcal{F}_i$  be the set of all boxes in  $\mathcal{F}'$  that intersect every  $B_j$  for  $j \neq i$  but do not intersect  $B_i$ . Let  $\mathcal{F}^*$  be the set of all boxes of  $\mathcal{F}'$  that intersect at most  $d - 2$  of the  $B_i$  boxes. As this is a partition of  $\mathcal{F}'$  we have

$$|\mathcal{F}_0| + \sum_{i=1}^d |\mathcal{F}_i| + |\mathcal{F}^*| = |\mathcal{F}'| = n - d.$$

Note that  $|\mathcal{F}_0| \leq k$  since every box in  $\mathcal{F}_0$  contains the point  $(c_1, \dots, c_d)$ .

Let  $N$  be the number of intersecting pairs between  $\{B_1, \dots, B_d\}$  and  $\mathcal{F}'$ . Each  $B_i$  intersects at least  $\deg B_i - (d - 1)$  boxes from  $\mathcal{F}'$  as  $B_i$  may intersect  $B_j$  for all  $j \in [d], j \neq i$ . Since every  $\deg B_i \geq (1 - \frac{1}{d})n + \frac{2k+1}{2d}$  we have

$$d \left( \left(1 - \frac{1}{d}\right)n + \frac{2k+1}{2d} - (d - 1) \right) \leq N$$

Every box in  $\mathcal{F}_0$  intersects every  $B_i$ ,  $i \in [d]$ , every box in  $\mathcal{F}_i$  intersects every  $B_j$  except for  $B_i$  and every box in  $\mathcal{F}^*$  intersects at most  $(d - 2)$  of the  $B_i$ . Consequently

$$N \leq d|\mathcal{F}_0| + (d - 1) \sum_{i=1}^d |\mathcal{F}_i| + (d - 2)|\mathcal{F}^*|.$$

So we have

$$\begin{aligned}
 d \left( \left(1 - \frac{1}{d}\right)n + \frac{2k+1}{2d} - (d-1) \right) &\leq d|\mathcal{F}_0| + (d-1) \sum_{i=1}^d |\mathcal{F}_i| + (d-2)|\mathcal{F}^*| \\
 &= |\mathcal{F}_0| + (d-1) \left( |\mathcal{F}_0| + \sum_{i=1}^d |\mathcal{F}_i| + |\mathcal{F}^*| \right) - |\mathcal{F}^*| \\
 &= |\mathcal{F}_0| + (d-1)(n-d) - |\mathcal{F}^*|.
 \end{aligned}$$

Simplifying the inequality and using  $|\mathcal{F}_0| \leq k$  give

$$k + \frac{1}{2} \leq |\mathcal{F}_0| - |\mathcal{F}^*| \leq k - |\mathcal{F}^*|$$

implying  $|\mathcal{F}^*| \leq -\frac{1}{2}$ , which is a contradiction.

**2.2. Proof of Corollary 1.** If no point of  $\mathbb{R}^d$  belongs to  $dn\varepsilon - \frac{d}{2} + 1$  elements of  $\mathcal{F}$ , then by Theorem 1 the number of intersecting pairs of  $\mathcal{F}$  is smaller than

$$\frac{d-1}{2d}n^2 + \frac{2(dn\varepsilon - \frac{d}{2}) + d}{2d}n = \left( \frac{d-1}{2d} + \varepsilon \right) n^2,$$

which yields a contradiction.

**2.3. Proof of Theorem 2.** Let  $\pi_i$  denote the orthogonal projection to the  $i$ th dimension in  $\mathbb{R}^d$ , that is,  $\pi_i(B) = (a_i(B), b_i(B))$  for  $B \in \mathcal{F}$ . Set  $\varepsilon = 1 - \alpha$ . Define  $T_i = \{\pi_i(B) : B \in \mathcal{F}\}$ ; this is a family of  $n$  intervals, and all but at most  $\varepsilon \binom{n}{2}$  of the pairs in  $T_i$  intersect. According to the sharp version of the fractional Helly theorem (cf. [6]),  $T_i$  contains an intersecting subfamily  $T'_i$  of size  $(1 - \sqrt{\varepsilon})n$ , let  $c_i$  be a common point of all the intervals in  $T'_i$ . Define  $D_i = \{B \in \mathcal{F} : c_i \notin \pi_i(B)\}$ . Then  $\mathcal{F} \setminus \bigcup_1^d D_i$  consists of at least  $(1 - d\sqrt{\varepsilon})n = (1 - d\sqrt{1-\alpha})n$  boxes and all of them contain the point  $(c_1, \dots, c_d)$ .

**2.4. Proof of Proposition 1.** Let  $k \in \{1, \dots, n\}$  be an integer, and let  $\mathcal{F}$  be the family of open intervals  $(i, i+k)$  for  $i = 1, 2, \dots, n$ . Thus  $\mathcal{F}$  consists of  $n$  intervals, no  $k+1$  of them have a point in common, and there are  $(k-1)n - \binom{k}{2}$  intersecting pairs in  $\mathcal{F}$ . Consequently  $T(n, k, 1) \geq (k-1)n - \binom{k}{2}$ .

Next we show, by induction on  $n$  that  $T(n, k, 1) \leq (k-1)n - \binom{k}{2}$ . Let  $\mathcal{F}$  be a family of  $n$  intervals such that no  $k+1$  of them have a common point. We assume that these intervals are closed which is no loss of generality. The statement is clearly true when  $n = k$ . Let  $[a, b] \in \mathcal{F}$  be the interval where  $b$  is minimal. Since any interval intersecting  $[a, b]$  contains  $b$ , there are at most  $k-1$  intervals intersecting  $[a, b]$ . Removing  $[a, b]$  from  $\mathcal{F}$  and applying induction, we find there are at most  $(k-1)(n-1) - \binom{k}{2}$  intersecting pairs in  $\mathcal{F} \setminus \{[a, b]\}$ . That is, there are at most  $k-1 + (k-1)(n-1) - \binom{k}{2} = (k-1)n - \binom{k}{2}$  intersecting pairs in  $\mathcal{F}$ .

### 3. ACKNOWLEDGEMENTS

The authors wish to acknowledge the support of this research by the Hungarian-Mexican Intergovernmental S&T Cooperation Programme grant TÉT\_10-1-2011-0471 and NIH B330/479/11 “Discrete and Convex Geometry”. The first and the last authors were partially supported by ERC Advanced Research Grant no. 267165 (DISCONV), and the first author by Hungarian National Research Grant K 83767,

as well. The second author was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences. The third author was partially supported by MTM 2012-30719. The fourth and fifth authors acknowledge partial support from CONACyT under project 166306 and PAPIIT IN101912.

## REFERENCES

- [1] Ludwig Danzer, Branko Grünbaum, and Victor Klee, *Helly's theorem and its relatives*, Proc. Sympos. Pure Math., Vol. VII, Amer. Math. Soc., Providence, R.I., 1963, pp. 101–180.
- [2] Ludwig Danzer and Branko Grünbaum, *Intersection properties of boxes in  $\mathbf{R}^d$* , *Combinatorica* **2** (1982), no. 3, 237–246.
- [3] Reinhard Diestel, *Graph theory*, 4th ed., Graduate Texts in Mathematics, vol. 173, Springer, Heidelberg, 2010.
- [4] Jürgen Eckhoff, *A survey of the Hadwiger-Debrunner  $(p, q)$ -problem*, Discrete and computational geometry, Algorithms Combin., vol. 25, Springer, Berlin, 2003, pp. 347–377.
- [5] Jürgen Eckhoff, *Helly, Radon, and Carathéodory type theorems*, Handbook of convex geometry, Vol. A, B, North-Holland, Amsterdam, 1993, pp. 389–448.
- [6] Gil Kalai, *Intersection patterns of convex sets*, Israel J. Math. **48** (1984), no. 2-3, 161–174.
- [7] M. Katchalski and A. Liu, *A problem of geometry in  $\mathbf{R}^n$* , Proc. Amer. Math. Soc. **75** (1979), no. 2, 284–288.
- [8] Jiří Matoušek, *Lectures on discrete geometry*, Graduate Texts in Mathematics, vol. 212, Springer-Verlag, New York, 2002.

ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS, PO BOX 127, H-1364 BUDAPEST, HUNGARY,  
AND DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE LONDON, GOWER STREET, LONDON,  
WC1E 6BT, U.K.

*E-mail address:* `barany@renyi.hu`

DEPARTMENT OF GEOMETRY, BOLYAI INSTITUTE, UNIVERSITY OF SZEGED, ARADI VÉRTANÚK  
TERE 1, H-6720 SZEGED, HUNGARY, AND DEPARTMENT OF MATHEMATICS AND STATISTICS, UNI-  
VERSITY OF CALGARY, CANADA

*E-mail address:* `fodorf@math.u-szeged.hu`

UNIVERSIDAD DE CASTILLA- LA MANCHA DEPARTAMENTO DE ANÁLISIS ECONÓMICO Y FINANZAS.  
UNIVERSIDAD DE CASTILLA-LA MANCHA. AVDA. REAL FÁBRICA DE SEDA, s/n. 45600 TALAVERA  
DE LA REINA. TOLEDO. SPAIN.

*E-mail address:* `alvaro.martinezperez@uclm.es`

INSTITUTO DE MATEMÁTICAS, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, ÁREA DE LA  
INVESTIGACIÓN CIENTÍFICA, CIRCUITO EXTERIOR, CU. COYOACAN 04510, MÉXICO D.F., MÉXICO

*E-mail address:* `luis@matem.unam.mx`

INSTITUTO DE MATEMÁTICAS, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, ÁREA DE LA  
INVESTIGACIÓN CIENTÍFICA, CIRCUITO EXTERIOR, CU. COYOACAN 04510, MÉXICO D.F., MÉXICO

*E-mail address:* `dolivero@matem.unam.mx`

DEPARTMENT OF MATHEMATICS, WESTERN KENTUCKY UNIVERSITY, BOWLING GREEN, KY  
42101, USA

*E-mail address:* `attila.por@wku.edu`